TECHNISCHE UNIVERSITÄT
KAISERSLAUTERN

SCHRIFTEN ZUR

FUNKTIONALANALYSIS UND GEOMATHEMATIK

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Locally Supported Approximate Identities on the Unit Ball

Bericht 30 – Dezember 2006

FACHBEREICH MATHEMATIK

Locally Supported Approximate Identities on the Unit Ball

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March 26, 2007

Abstract

We present a theory for locally supported approximate identities on the unit ball in $\mathbb{R}^3$. The uniform convergence of the convolutions of the derived kernels with an arbitrary continuous function f to f, i.e. the defining property of an approximate identity, is proved. Moreover, a closed representation for a class of such kernels is given.

Key Words: local support, approximate identity, ball, uniform convergence, convolution, closed representation.

AMS(2000) Classification: 41A30, 41A35, 41A63, 44A35, 86-08

1 Introduction

The advantages of locally supported kernels for approximating structures are well–known and have already been studied intensively also for the sphere. The early paper [15] uses a tensor product ansatz to construct locally supported splines on the sphere. In [2, 6] axisymmetric locally supported basis functions on the 2–sphere are introduced by use of Bernstein polynomials. Their treatment as foundation for spherical spline interpolation in certain Sobolev spaces is discussed in [14]. In [5] their application to satellite data is investigated. Their iterated convolutions yields further advantages such as a multiresolution analysis and locally supported spherical wavelets, see the construction of spherical up functions [7]. This topic is also discussed in [1] where additionally a closed representation of the iterated kernels is derived. In [8] locally supported wavelets and multiresolution analyses are developed for a general manifold setting with special consideration of the sphere. Furthermore, [12] treats among others locally supported splines on

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the n–sphere based on Wendland's function.
In this paper we show how locally supported kernels on the 3–dimensional unit ball can be constructed. We prove that those kernels establish an approximate identity for all continuous functions on the ball. The corresponding convergence is uniform.
Moreover, we derive a closed representation for a certain family of such locally supported approximate identities allowing the choice of the smoothness of the kernel function. Without closed representation of the kernel Φ_δ one would have to do a numerical integration over the intersection of two balls to evaluate the kernel, where each x requires such a numerical integration to get all values $\Phi_\delta(x, y)$. This can, certainly, be realised in practice. However, the derived closed representation essentially accelerates the calculation process and reduces the total error. Only the numerical calculation of the convolution integral of the approximate identity is left.
Note that approximate identities on balls in $\mathbb{R}^3$ are studied e.g. in [11]. Further works on localizing (but not necessarily locally supported) kernels, like scaling functions and wavelets, on the 3D ball can e.g. be found in [9, 10, 13].
An exemplary application of approximating structures on three–dimensional balls can be found in geophysics. There, the choice of appropriate tools for describing features of the Earth's interior, such as the mass density, the speed of propagation of seismic P and S waves and other rheological quantities, is still a field of research.

2 Definitions and Notation

$\mathbb{N}_0$ and $\mathbb{R}$ represent the sets of all non–negative integers and all real numbers, respectively, such that $\mathbb{R}^3$ is the 3-dimensional Euclidean space. Moreover, let $B := \left\{x \in \mathbb{R}^3 \mid \ |x| \leq 1\right\}$ be the closed unit ball in $\mathbb{R}^3$ with centre at the origin, where $|x| := \sqrt{\sum_{i=1}^3 x_i^2}$ is the Euclidean norm in $\mathbb{R}^3$, and $B_\delta(x) := \left\{y \in \mathbb{R}^3 \mid \ |x - y| < \delta\right\}$, $\delta > 0$, be the open ball of radius δ with centre at x in $\mathbb{R}^3$. As usual, $\mathrm{C}(B)$ denotes the Banach space of all real–valued continuous functions on B with the norm $\|f\|_{\mathrm{C}(B)} := \sup_{x\in B} |f(x)|$, $f \in \mathrm{C}(B)$.
Let $\Phi : B \times B \to \mathbb{R}$ be an integrable function such that for each $x \in B$, $\Phi(x, \cdot)$ is positive almost everywhere in B. For $0 < \delta < 2$ and for all $x \in B$ we define $\Psi_\delta(x) := \int_{B\cap B_\delta(x)} \Phi(x, z)\, dz$. With this abbreviation the kernel

$\Phi_\delta : B \times B \to \mathbb{R}$ is defined by

$$\Phi_\delta(x,y) := \begin{cases} \frac{\Phi(x,y)}{\Psi_\delta(x)} & \text{if } y \in B \cap B_\delta(x) \\ 0 & \text{otherwise} \end{cases} . \tag{1}$$

Obviously, Φ_δ has the local support $\operatorname{supp}(\Phi_\delta(x,\cdot)) = \overline{B \cap B_\delta(x)}$. Note that, usually, we choose the parameter $\delta > 0$ sufficiently close to 0.
Moreover, for $f \in \mathrm{C}(B)$ we define the convolution of f and Φ_δ by

$$f \star \Phi_\delta := \int_B f(y)\Phi_\delta(\cdot, y)\, dy.$$

3 Approximate Identity

Theorem 3.1 *Let Φ and Φ_δ be as defined above. If f is a continuous function defined on B then $\|f \star \Phi_\delta - f\|_{\mathrm{C}(B)} \to 0$ as $\delta \to 0+$.*

Proof. From the construction of Φ_δ, it is clear that

$$\int_B \Phi_\delta(x,y)dy = \int_{B\cap B_\delta(x)} \Phi_\delta(x,y)dy = 1 \tag{2}$$

for all $x \in B$ and all $\delta > 0$. Therefore, we can write

$$\begin{aligned} |(f \star \Phi_\delta)(x) - f(x)| &= \left| \int_B (f(y) - f(x))\ \Phi_\delta(x,y)\, dy \right| \\ &\leq \int_{B\cap B_\delta(x)} |f(y) - f(x)|\ \Phi_\delta(x,y)\, dy \end{aligned}$$

Since f is continuous on the compact set B, it is uniformly continuous on B, therefore, for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|f \star \Phi_\delta - f\|_{\mathrm{C}(B)} \leq \sup_{x\in B} \left(\int_{B\cap B_\delta(x)} \varepsilon\ \Phi_\delta(x,y)\, dy \right) = \varepsilon$$

by (2). This also holds true if we replace δ by $\tilde{\delta} \in]0,\delta]$. Hence, as $\delta \to 0+$ we have $\|f \star \Phi_\delta - f\|_{\mathrm{C}(B)} \to 0$. ∎

Now we present some examples of locally supported approximate identities.

Example 3.1 *The function* $\Phi_{\delta;k} : B \times B \to \mathbb{R}$ *defined by*

$$\Phi_{\delta;k}(x,y) := \begin{cases} (\delta - |x-y|)^k \Big(\int_{B \cap B_\delta(x)} (\delta - |x-z|)^k \, dz \Big)^{-1} & \text{if } y \in B \cap B_\delta(x) \\ 0 & \text{otherwise} \end{cases}$$

for all $k \in \mathbb{N}_0$*, is a locally supported approximate identity in* B*. Note that* $\Phi_{\delta;k}(x,\cdot) \in \mathrm{C}^{(k-1)}(B \setminus \{x\}) \cap \mathrm{C}(B)$ *for* $k \geq 1$*. We will derive a closed representation for this family of kernels in the next section.*

Example 3.2 *If we define* $\Phi(x,y) = \exp(-k|x-y|)$ *for all* $x, y \in B$ *and* $k \in \mathbb{N}_0$ *then the function in (1) is also a locally supported approximate identity.*

4 Closed Representation of Some Kernels

We are interested here to calculate expressions of the form

$$\int_{B \cap B_\delta(x)} (\delta - |x-z|)^k \, dz = \sum_{j=0}^{k} \binom{k}{j} \delta^{k-j} (-1)^j \int_{B \cap B_\delta(x)} |x-z|^j \, dz.$$

We, thus, restrict our attention to the determination of

$$F_j(x) := \int_{B \cap B_\delta(x)} |x-z|^j \, dz.$$

For symmetry reasons it suffices to assume that $x = (0,0,|x|)^{\mathrm{T}}$. Note that the intersection of $\partial B_\delta(x)$ and ∂B is a planar circle whose plane cuts $B_\delta(x)$. We have to distinguish two cases:

(i) x is below or on this cutting plane.

(ii) x is above this cutting plane.

In each case we subdivide the integral into the integration over three different domains, see Figures 1 and 2.
Obviously, we have in case (i)

$$F_j(x) = \int_{P_1} |x-z|^j \, dz + \int_{P_2} |x-z|^j \, dz + \int_{P_3} |x-z|^j \, dz$$

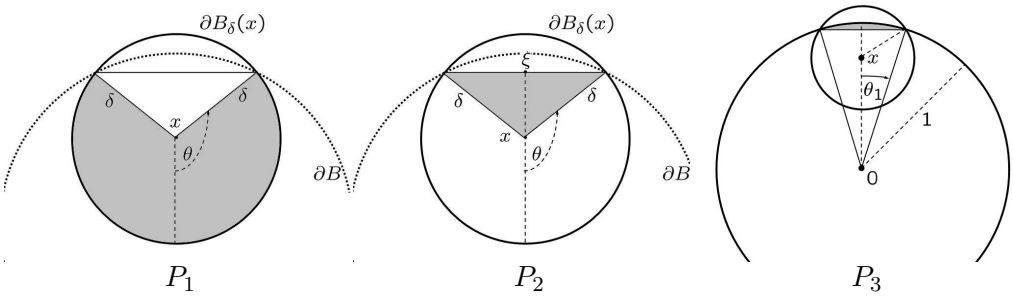

Figure 1: Subdivision of the integration in case (i)

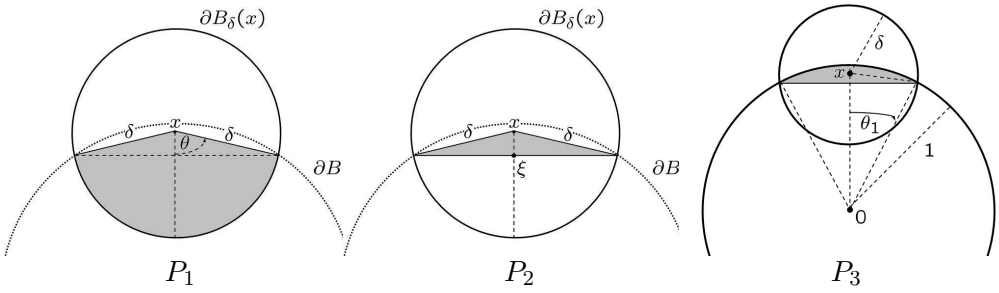

Figure 2: Subdivision of the integration in case (ii)

and in case (ii)

$$F_j(x) = \int_{P_1} |x-z|^j \, dz - \int_{P_2} |x-z|^j \, dz + \int_{P_3} |x-z|^j \, dz.$$

We use the following parameterization in polar coordinates

$$z = x + \begin{pmatrix} r\sin\vartheta\cos\varphi \\ r\sin\vartheta\sin\varphi \\ r\cos\vartheta \end{pmatrix}. \tag{3}$$

For P_1 we have the ranges $r \in [0,\delta]$, $\varphi \in [0, 2\pi[$, $\vartheta \in [\pi-\theta, \pi]$. The angle θ can be calculated by the cosine theorem as follows:

$$1^2 = |x|^2 + \delta^2 - 2|x|\delta\cos\theta \quad \Leftrightarrow \quad \cos\theta = \frac{|x|^2+\delta^2-1}{2|x|\delta} \ . \tag{4}$$

Note that $\delta > 0$ and that the case $x = 0$ corresponds to $\theta = \pi$ for sufficiently small δ. Thus, using $|x - z| = r$ we obtain

$$\begin{aligned}\int_{P_1} |x-z|^j \, dz &= \int_0^\delta \int_0^{2\pi} \int_{\pi-\theta}^{\pi} r^j \cdot r^2 \sin\vartheta \, d\vartheta \, d\varphi \, dr \\ &= \frac{2\pi}{j+3} \, \delta^{j+3} \, (-\cos\vartheta)|_{\pi-\theta}^{\pi} = \frac{2\pi}{j+3} \, \delta^{j+3}(1 + \cos(\pi - \theta)) \\ &= \frac{2\pi}{j+3} \, \delta^{j+3}(1 - \cos\theta).\end{aligned}$$

Note that this result is true in cases (i) and (ii) and is also valid if $B_\delta(x) \subset B$ where $\theta = \pi$. Hence,

$$\int_{P_1} |x-z|^j \, dz = \begin{cases} \frac{4\pi}{j+3} \, \delta^{j+3} & \text{if } |x| + \delta \le 1 \\ \frac{2\pi}{j+3} \, \delta^{j+3} \left(1 - \frac{|x|^2+\delta^2-1}{2|x|\delta}\right) & \text{else.} \end{cases}$$

Obviously, in case of $|x| + \delta \le 1$, in particular $x = 0$, the other two integrals are not needed.
P_2 represents a cone. We parameterise it again using (3) but in case (i) with the ranges $\varphi \in [0, 2\pi[$, $\vartheta \in [0, \pi - \theta]$, and $r \in [0, \rho(\vartheta)]$ where

$$\cos\vartheta = \frac{|\xi| - |x|}{\rho(\vartheta)} \quad \Leftrightarrow \quad \rho(\vartheta) = \frac{|\xi| - |x|}{\cos\vartheta} \ .$$

Introducing the angle $\theta_1 = \arccos|\xi|$ (see Figures 1 and 2) we get from the cosine theorem

$$\delta^2 = |x|^2 + 1^2 - 2|x| \cdot 1 \cdot \cos\theta_1 \quad \Leftrightarrow \quad (|\xi| =) \cos\theta_1 = \frac{|x|^2 + 1 - \delta^2}{2|x|} \ .$$

Hence,

$$\rho(\vartheta) = \frac{1}{\cos\vartheta} \, \frac{-|x|^2 + 1 - \delta^2}{2|x|} \ .$$

Note that the case $\vartheta = \pi/2$ can only occur if $\theta = \pi/2$ which corresponds to $|\xi| = |x|$ where $\mathrm{Vol}\,(P_2) = 0$.
In case (ii), the ranges are $\varphi \in [0, 2\pi[$, $\vartheta \in [\pi - \theta, \pi]$, and $r \in [0, \tilde\rho(\vartheta)]$ with

$$\cos(\pi - \vartheta) = \frac{|x| - |\xi|}{\tilde\rho(\vartheta)} \quad \Leftrightarrow \quad -\cos\vartheta = \frac{|x| - |\xi|}{\tilde\rho(\vartheta)}$$

such that $\tilde{\rho}(\vartheta) = \rho(\vartheta)$.
Hence, writing in general $\vartheta \in [\vartheta_1, \vartheta_2]$ we get

$$\begin{aligned}
\int_{P_2} |x-z|^j \, dz &= \int_{\vartheta_1}^{\vartheta_2} \int_0^{\rho(\vartheta)} \int_0^{2\pi} r^j r^2 \sin\vartheta \, d\varphi \, dr \, d\vartheta \\
&= \frac{2\pi}{j+3} \int_{\vartheta_1}^{\vartheta_2} \rho(\vartheta)^{j+3} \sin\vartheta \, d\vartheta \\
&= \frac{2\pi}{j+3} \left(\frac{1-\delta^2-|x|^2}{2|x|} \right)^{j+3} \int_{\vartheta_1}^{\vartheta_2} \frac{\sin\vartheta}{\cos^{j+3}\vartheta} \, d\vartheta \\
&= \frac{2\pi}{j+3} \left(\frac{1-\delta^2-|x|^2}{2|x|} \right)^{j+3} \int_{\cos\vartheta_1}^{\cos\vartheta_2} \frac{-1}{t^{j+3}} \, dt \\
&= \frac{2\pi}{(j+3)(j+2)} \left(\frac{1-\delta^2-|x|^2}{2|x|} \right)^{j+3} \left(\frac{1}{\cos^{j+2}\vartheta_2} - \frac{1}{\cos^{j+2}\vartheta_1} \right).
\end{aligned}$$

Hence, using (4) we arrive at

$$\begin{aligned}
\int_{P_2} |x-z|^j \, dz = \frac{2\pi}{(j+3)(j+2)} \left(\frac{1-\delta^2-|x|^2}{2|x|} \right)^{j+3} \\
\times \begin{cases} \left(\left(\frac{1-|x|^2-\delta^2}{2|x|\delta} \right)^{-j-2} - 1 \right) & \text{in case (i)} \\ \left((-1)^{j+2} - \left(\frac{1-|x|^2-\delta^2}{2|x|\delta} \right)^{-j-2} \right) & \text{in case (ii).} \end{cases}
\end{aligned}$$

Note that the two cases are characterised as follows:

(i) $|\xi| \geq |x| \Leftrightarrow \theta \geq \frac{\pi}{2} \Leftrightarrow |x|^2 + 1 - \delta^2 \geq 2|x|^2 \Leftrightarrow 1 - \delta^2 \geq |x|^2$

(ii) $|\xi| < |x| \Leftrightarrow \theta < \frac{\pi}{2} \Leftrightarrow 1 - \delta^2 < |x|^2$

In this sense, the case $\theta = \frac{\pi}{2} \Leftrightarrow |x|^2 = 1 - \delta^2$, where the integral vanishes, is also covered by the final formula.
P_3 is parameterised by (3), too. The ranges are in case (i) given by $\varphi \in [0, 2\pi[$, $\vartheta \in [0, \pi - \theta]$, and $r \in [\rho(\vartheta), R(\vartheta)]$, where $\rho(\vartheta)$ is the same expression that was used for P_2. The larger radius referring to ∂B is given by the cosine theorem (see Figure 3)

$$1^2 = |x|^2 + R(\vartheta)^2 - 2|x|R(\vartheta)\cos(\pi - \vartheta) \Leftrightarrow 0 = R(\vartheta)^2 + 2|x|R(\vartheta)\cos\vartheta + |x|^2 - 1.$$

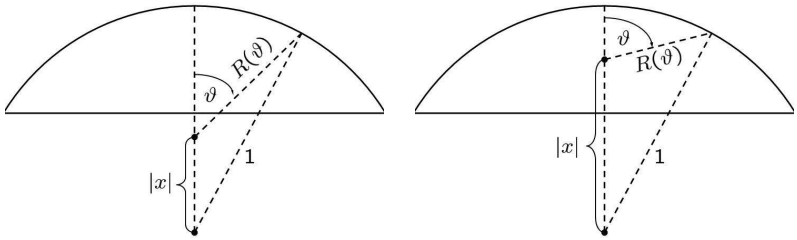

Figure 3: Illustration of the quantities, left: case (i), right: case (ii).

This equation has only one positive solution for $R(\vartheta)$, namely

$$R_1(\vartheta) = -|x|\cos\vartheta + \sqrt{|x|^2\cos^2\vartheta - |x|^2 + 1} = -|x|\cos\vartheta + \sqrt{-|x|^2\sin^2\vartheta + 1}.$$

In case (ii) we have $\varphi \in [0, 2\pi[$, $\vartheta \in [0, \pi]$, and $r \in [0, R_2(\vartheta)]$ where

$$R_2(\vartheta) = \begin{cases} -|x|\cos\vartheta + \sqrt{-|x|^2\sin^2\vartheta + 1} & \text{if } \vartheta \le \pi - \theta \\ \rho(\vartheta) & \text{if } \vartheta > \pi - \theta. \end{cases}$$

Note that $R_2(\pi - \theta) = \delta$ and $\rho(\pi - \theta) = \delta$.
For case (i) we obtain

$$\begin{aligned} \int_{P_3} |x - z|^j \, dz &= \int_0^{2\pi} \int_0^{\pi-\theta} \int_{\rho(\vartheta)}^{R_1(\vartheta)} r^{j+2} \sin\vartheta \, dr \, d\vartheta \, d\varphi \\ &= 2\pi \int_0^{\pi-\theta} \frac{1}{j+3} \left(R_1(\vartheta)^{j+3} - \rho(\vartheta)^{j+3} \right) \sin\vartheta \, d\vartheta \\ &= \frac{2\pi}{j+3} \int_0^{\pi-\theta} \left(\left(-|x|\cos\vartheta + \sqrt{-|x|^2\sin^2\vartheta + 1} \right)^{j+3} \right. \\ &\qquad \left. - \left(\frac{1 - |x|^2 - \delta^2}{2|x|} \right)^{j+3} \frac{1}{\cos^{j+3}\vartheta} \right) \sin\vartheta \, d\vartheta. \end{aligned}$$

With the substitution $t = |x|\cos\vartheta$ we obtain

$$\begin{aligned} \int_{P_3} |x - z|^j \, dz = -\frac{2\pi}{j+3} \int_{|x|}^{-|x|\cos\theta} \left(\left(-t + \sqrt{t^2 - |x|^2 + 1} \right)^{j+3} \right. \\ \left. - 2^{-j-3} \left(1 - |x|^2 - \delta^2 \right)^{j+3} \frac{1}{t^{j+3}} \right) \frac{1}{|x|} \, dt \end{aligned}$$

$$= \frac{2\pi}{|x|(j+3)} \sum_{r=0}^{j+3} \binom{j+3}{r} \int_{-|x|\cos\theta}^{|x|} (-t)^r \left(t^2 - |x|^2 + 1\right)^{\frac{j+3-r}{2}} dt$$

$$+ \frac{2\pi}{|x|(j+3)} 2^{-j-3} \left(1 - |x|^2 - \delta^2\right)^{j+3} \frac{1}{-(j+2)} \left((-|x|\cos\theta)^{-j-2} - |x|^{-j-2}\right).$$

We use the abbreviation

$$I_{j,r}(|x|, -|x|\cos\theta) := \int_{-|x|\cos\theta}^{|x|} t^r \left(t^2 - |x|^2 + 1\right)^{\frac{j+3-r}{2}} dt.$$

Using [3], p. 49, we get

a) if r is even and $j+3-r$ is odd, i.e. j is even, then

$$I_{j,r}(|x|, -|x|\cos\theta) = \left(\chi_{\mathbb{R}^+}(r) \frac{(t^2+1-|x|^2)^{\frac{j+5-r}{2}}}{j+4} \left(t^{r-1} \right.\right.$$

$$\left. + \sum_{q=1}^{\frac{r}{2}-1} (-1)^q \frac{(r-1)(r-3)\cdots(r-2q+1)}{(j+2)j\cdots(j-2q+4)} \left(1-|x|^2\right)^q t^{r-2q-1} \right)$$

$$+ (-1)^{\frac{r}{2}} (r-1)!! \left(1-|x|^2\right)^{\frac{r}{2}} \left(\prod_{p=-\frac{r}{2}+3}^{2} (j+2p) \right)^{-1}$$

$$\times \left(\frac{t\left(t^2+1-|x|^2\right)^{\frac{1}{2}}}{j-r+4} \left(\left(t^2+1-|x|^2\right)^{\frac{j-r}{2}+1} + \right.\right.$$

$$+ \sum_{q=1}^{\frac{j-r}{2}+1} \frac{(j-r+3)(j-r+1)\cdots(j-r-2q+5)}{(j-r+2)(j-r)\cdots(j-r-2q+4)}$$

$$\left. \left. \times \left(1-|x|^2\right)^q \left(t^2+1-|x|^2\right)^{\frac{j-r}{2}+1-q} \right) \right.$$

$$\left. \left. + \frac{(j-r+3)!! \left(1-|x|^2\right)^{\frac{j-r}{2}+2}}{2^{\frac{j-r}{2}+2}\left(\frac{j-r}{2}+2\right)!} \ln\left(t + \left(t^2+1-|x|^2\right)^{\frac{1}{2}}\right) \right) \right) \Bigg|_{-|x|\cos\theta}^{|x|} \quad (5)$$

where $(2p+1)!! := 1 \cdot 3 \cdot 5 \cdots (2p+1)$ for $p \in \mathbb{N}_0$. It should be noted here that empty sums are set to zero (i.e. $\sum_{\nu=a}^{b} \ldots := 0$ if $b < a$),

empty products are set to one, $\chi_{\mathbb{R}^+}$ represents the indicator function of $\mathbb{R}^+$, and $(-1)!! := 1$.

b) if r is odd and $j+3-r$ is odd, i.e. j is odd, then

$$I_{j,r}(|x|,-|x|\cos\theta) = \sum_{q=0}^{\frac{r-1}{2}} \binom{\frac{r-1}{2}}{q} \frac{\left(|x|^2-1\right)^{\frac{r-1}{2}-q}\left(t^2+1-|x|^2\right)^{q+\frac{j+5-r}{2}}}{j-r+5+2q}\Bigg|_{-|x|\cos\theta}^{|x|}. \tag{6}$$

For even $j+3-r$ the integration is easy:

$$I_{j,r}(|x|,-|x|\cos\theta) = \sum_{q=0}^{\frac{j+3-r}{2}} \binom{\frac{j+3-r}{2}}{q} \int_{-|x|\cos\theta}^{|x|} t^{r+2q}\,dt\,\left(1-|x|^2\right)^{\frac{j+3-r}{2}-q} \tag{7}$$

$$= \sum_{q=0}^{\frac{j+3-r}{2}} \binom{\frac{j+3-r}{2}}{q} \frac{1}{r+2q+1}\left(|x|^{r+2q+1}-(-|x|\cos\theta)^{r+2q+1}\right)\left(1-|x|^2\right)^{\frac{j+3-r}{2}-q}.$$

Hence,

$$\int_{P_3}|x-z|^j\,dz = \frac{2\pi}{|x|(j+3)}\sum_{r=0}^{j+3}\binom{j+3}{r}(-1)^r I_{j,r}\left(|x|,\frac{1-|x|^2-\delta^2}{2\delta}\right)$$
$$-\frac{\pi 2^{-j-2}}{|x|(j+3)(j+2)}\left(1-|x|^2-\delta^2\right)^{j+3}\left(\left(\frac{1-|x|^2-\delta^2}{2\delta}\right)^{-j-2}-|x|^{-j-2}\right)$$

in case (i). In case (ii) we obtain

$$\begin{aligned}\int_{P_3}|x-z|^j\,dz &= \int_0^{2\pi}\int_0^{\pi-\theta}\int_0^{R_1(\vartheta)} r^{j+2}\sin\vartheta\,dr\,d\vartheta\,d\varphi \\ &\quad + \int_0^{2\pi}\int_{\pi-\theta}^{\pi}\int_0^{\rho(\vartheta)} r^{j+2}\sin\vartheta\,dr\,d\vartheta\,d\varphi \\ &= -\frac{2\pi}{j+3}\int_{|x|}^{-|x|\cos\theta}\left(-t+\sqrt{t^2-|x|^2+1}\right)^{j+3}\frac{1}{|x|}\,dt + \int_{P_2}|x-z|^j\,dz \\ &= \frac{2\pi}{|x|(j+3)}\sum_{r=0}^{j+3}\binom{j+3}{r}(-1)^r\int_{-|x|\cos\theta}^{|x|} t^r\left(t^2-|x|^2+1\right)^{\frac{j+3-r}{2}}\,dt\end{aligned}$$

$$
\begin{aligned}
&+\int_{P_2}|x-z|^j\,dz\\
=\;&\frac{2\pi}{|x|(j+3)}\sum_{k=0}^{j+3}\binom{j+3}{k}(-1)^r I_{j,r}(|x|,-|x|\cos\theta)+\int_{P_2}|x-z|^j\,dz.
\end{aligned}
$$

We, thus, have a closed representation for the kernels

$$
\Phi_{\delta;k}(x,y):=\begin{cases}(\delta-|x-y|)^k\left(\int_{B\cap B_\delta(x)}(\delta-|x-z|)^k\,dz\right)^{-1} & \text{if } y\in B\cap B_\delta(x)\\ 0 & \text{otherwise,}\end{cases}
\tag{8}
$$

$k\in\mathbb{N}_0$. We summarize here the formula for the denominator in (8):

$$
\int_{B\cap B_\delta(x)}(\delta-|x-z|)^k\,dz=\sum_{j=0}^{k}\binom{k}{j}\delta^{k-j}(-1)^j\,\frac{4\pi}{j+3}\,\delta^{j+3}\quad\text{if } |x|+\delta\le 1,
$$

$$
\begin{aligned}
\int_{B\cap B_\delta(x)}(\delta-|x-z|)^k\,dz=&\sum_{j=0}^{k}\binom{k}{j}\delta^{k-j}(-1)^j\left(\frac{2\pi}{j+3}\,\delta^{j+3}\left(1-\frac{|x|^2+\delta^2-1}{2|x|\delta}\right)\right.\\
&+\frac{2\pi}{(j+3)(j+2)}\left(\frac{1-\delta^2-|x|^2}{2|x|}\right)^{j+3}\left(\left(\frac{1-|x|^2-\delta^2}{2|x|\delta}\right)^{-j-2}-1\right)\\
&+\frac{2\pi}{|x|(j+3)}\sum_{r=0}^{j+3}\binom{j+3}{r}(-1)^r I_{j,r}\left(|x|,\frac{1-|x|^2-\delta^2}{2\delta}\right)\\
&\left.-\frac{\pi 2^{-j-2}}{|x|(j+3)(j+2)}\left(1-|x|^2-\delta^2\right)^{j+3}\left(\left(\frac{1-|x|^2-\delta^2}{2\delta}\right)^{-j-2}-|x|^{-j-2}\right)\right),\\
&\text{if } |x|+\delta>1 \text{ and } |x|^2\le 1-\delta^2,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{B\cap B_\delta(x)}(\delta-|x-z|)^k\,dz\\
&\quad=\sum_{j=0}^{k}\binom{k}{j}\delta^{k-j}(-1)^j\left(\frac{2\pi}{j+3}\,\delta^{j+3}\left(1-\frac{|x|^2+\delta^2-1}{2|x|\delta}\right)\right.\\
&\qquad\left.+\frac{2\pi}{|x|(j+3)}\sum_{r=0}^{j+3}\binom{j+3}{r}(-1)^r I_{j,r}\left(|x|,\frac{1-|x|^2-\delta^2}{2\delta}\right)\right)\\
&\qquad\text{if } |x|+\delta>1 \text{ and } |x|^2>1-\delta^2,
\end{aligned}
$$

where $I_{j,r}$ is given by (5), (6), and (7), respectively, depending on the parity of j and r.

5 Numerical Results

For numerical verification, we take a function as a test function and try to reconstruct it. To compute our convolution on B we use the standard separation of an integral on B

$$\int_B F(x)dx = \int_0^1 r^2 \int_{\partial B} F(r\xi)\, d\omega(\xi)\, dr.$$

For the integral over the sphere we used an equiangular grid and the quadrature method given in [4] and for the line integral we used Simpson's rule. Our test function is $f(y) = 250\sqrt{\frac{105}{2\pi}}(y_1^2 - y_2^2)y_3 \sin(50|y|)$, $y = (y_1, y_2, y_3) \in B$. We calculated the convolution of f with the approximate identity in Examples 3.1 (with $k = 1$) and 3.2 for $\delta = 0.01$ and $\delta = 0.004$ and plotted the result on the $y_1 = 0$ plane, see Figures 4 and 5. The kernel of Example 3.2 was calculated numerically.
For $\delta = 0.01$ we used $300 \times 300 \times 300$ grid points for the integration on B. We also calculated the rooted mean square error which is 6.0300 in case of the kernel of Example 3.1 and is 6.2460 in case of the kernel of Example 3.2 for this δ. For $\delta = 0.004$ we used $1000 \times 1600 \times 1600$ grid points for the numerical integration on B. The rooted mean square error is here 4.3077 in case of the kernel of Example 3.1 and 4.9977 in case of the kernel of Example 3.2 for this δ. Note that always only a small number of the grid points is relevant for each integral due to the local support.
In the plots of the error one can see that there is a significant error at the boundary but near the poles where there are more points the error is comparatively small. So one can reduce this by using a larger number of grid points near the boundary. We also observed that the kernel of Example 3.1 gives better results than the kernel of Example 3.2.

6 Acknowledgments

The authors gratefully acknowledge the financial support by the Higher Education Commission (HEC) Pakistan. Moreover, the authors thank Dipl.-Math. Paula Kammann for proofreading the manuscript.

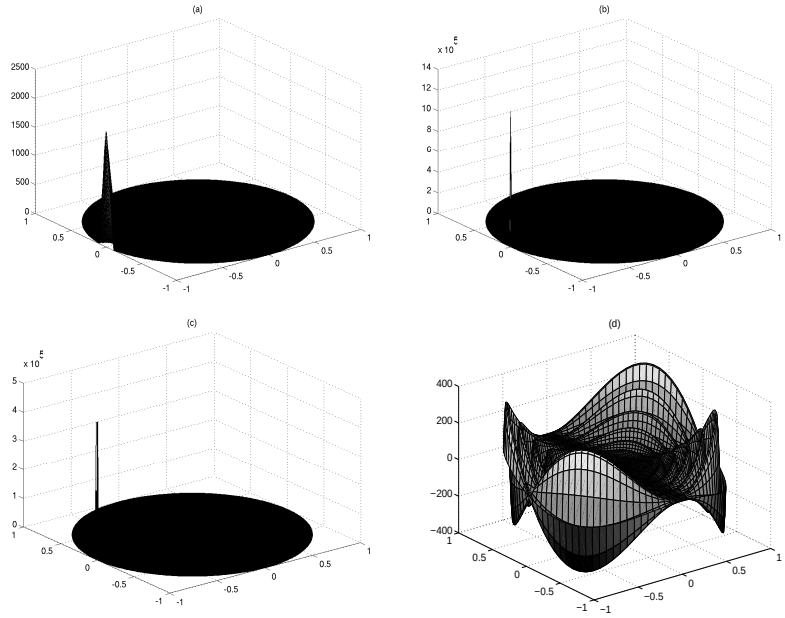

Figure 4: (a,b) Graph of the closed representation of the function given in Example 3.1, where $x = (0, -1, 0)$ is fixed, plotted in the $y_1 = 0$ plane. In (a) we have $\delta = 0.1$ and in (b) $\delta = 0.01$; (c) Graph of the function in Example 3.2, where $x = (0, -1, 0)$ is fixed, plotted in the $y_1 = 0$ plane with $\delta = 0.01$ and (d) is the graph of our test function

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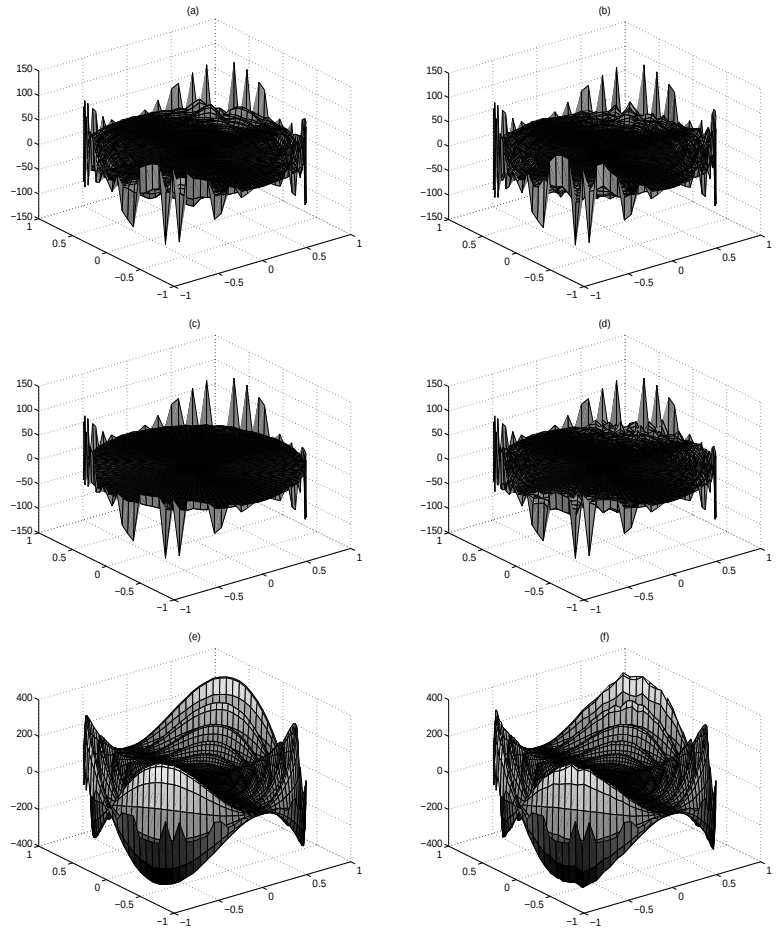

Figure 5: (a,c) are the graphs of the difference of the actual function f and the reconstructed function with parameter $\delta = 0.01$ and $\delta = 0.004$, respectively, using the kernel of Example 3.1 (with the closed representation), and (b,d) are the graphs of the difference of the actual function f and the reconstructed function with the same parameters as in (a,c), respectively, using the kernel of Example 3.2. (e) is the graph of the reconstructed function f with parameter $\delta = 0.004$ using the kernel of Example 3.1 and (f) is the graph of the reconstructed function f with parameter $\delta = 0.004$ using the kernel of Example 3.2.

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